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ANALYSING THE RESPONSES OF 7-8 YEAR OLDS WHEN SOLVING PARTITIONING PROBLEMS

KEY WORDS:, distribution problems, mathematical practice, mathematical representations, OSA approach, problem-solving, young children.

ABSTRACT

We analyze the mathematical solutions of 7-8 year old students when they individually solve an arithmetic problem. The analysis was based on the 'configuration of objects', an instrument derived from the onto-semiotic approach to mathematical knowledge -OSA. Results are illustrated through a number of cases. The aspects inferred from the overall analysis of the mathematical solutions include the use of iconic representations as a counting tool and the demonstrative nature of the arguments developed by students.

1. INTRODUCTION

Various studies have shown that young children are able to solve a variety of multiplication problems long before they are actually taught multiplication and division (Carpenter et al., 1993; Mulligan and Mitchelmore, 1997; Warfield, 2001; Bosch, Castro and Segovia, 2007). These studies highlight the importance of investigating the mathematical activity such children engaged in, especially as regards to the following questions: What representations do children use when solving problems? What strategies do they use? How does mathematical activity evolve in relation to age? How do the findings relate to the study of the introduction of the concepts of fraction and multiplication in the early years? What mistakes and difficulties emerge during the problem-solving process? These questions provide the focus for the present study.

One of the specific aspects addressed by research into how arithmetic problems are taught and learnt in the early years concerns the way in which pupils solve distribution problems. This broad category of problems includes situations in which a series of elements of one set can be separated, one by one, and redistributed among a variable number of participants. Here we examine the case in which the extra-mathematical situation requires the separation of elements of one set into groups. Yet, in the distribution of such elements, the whole is not a single (or several) unit(s) that must be divided into parts, instead, it implies the separation of the whole into discrete sets of various elements whose cardinal number may also be different.

Research into the distribution strategies used by young children has shown that pupils need to make visual-quantitative supports, which show visually-important quantitative aspects of the concepts involved in the solution of the problem (Fuson and Li, 2009). The role played by these representations in the problem-solving process forms the basis of our first research question:

1) What types of representations are used by 7-8 year olds when they individually solve distribution problems in an extra-mathematical context with verbal statements?

As these representations are merely the tip of an iceberg that also comprises calculation procedures and arguments, etc., we also formulated a second research question:

2) What configurations of procedures, arguments, propositions, concepts and representations are used by 7-8 year olds when they individually solve distribution problems in an extra-mathematical context with verbal statements? And what is the effect of the didactic contract on these configurations?

The paper is organized into six sections. Following this introduction the next section briefly reviews the literature on this topic. Section three then presents the theoretical and methodological tools that are used in the analysis, while section four describes the experimental design. This is followed in section five by the analysis and discussion of data concerning the first research question. Section six then presents the corresponding analysis and discussion of the second research question. The paper concludes with our final considerations.

2. REVIEW OF THE LITERATURE

As noted above, numerous studies have shown that pupils may solve a variety of multiplication problems long before they are taught about multiplication and division (Mulligan and Mitchelmore, 1997). Indeed, Carpenter, Ansell, Franke, Fennema & Weisbeck (1993) found that even kindergarten children could solve multiplication problems. Similar results were obtained by research studies carried out with children with special needs (Nunes, Bryant, Burman, Bell, Evans & Hallett, 2009).

Studies into the strategies used by young children when solving distribution problems, in which a discrete number of elements, that can be separated one by one, must be redistributed among a variable number of participants are numerous. Among them, we could cite; e.g., those in which the problem implied sharing out a certain number of biscuits among a group of children by means of a question that has multiple solutions (Davis and Hunting, 1990). Research has also focused on problems around the idea of dividing and distributing units. For example, Charles and Nason (2000), in a study that looked at the development of the fraction concept among eight-year-olds, proposed a type of problem in which the unit (or units) had to be divided into parts.

However, our review of the literature revealed only a few studies that examined the case in which the context requires the distribution of the whole into discrete sets of various elements whose cardinal number may also be different. Examples of such studies are those of Saundry and Nicol (2006) and of Bosch, Castro & Segovia (2007). They examined this type of problem with very young children.

In cognitive-based arithmetic problem-solving models, the situational component, not just the mathematical component, plays a significant role in the problem solving process, as it allows children to model the mathematical problem they need to solve. Such identification supports the mathematical activity they engage. Kintsch and Greeno (1985), among others, argue that a qualitative representation of all the elements involved in the problem, including both the characters and their intentions and the temporal and causal structure of the situation depicted in the problem. Such representation would emerge between the original text and the problem model. This idea is also supported by Reusser (1988) when they claim that in order to solve an arithmetic problem it is necessary to create a situational episodic model.

Several authors (e.g., Saundry and Nicol, 2006; Vicente, Orrantia & Verschaffel, 2008; Edo, Planas & Badillo, 2009) have studied, from different perspectives, the drawings used by children in the mathematics classroom. Smith (2003) and Woleck (2001) stated that pupils' drawings fulfil two basic functions. Firstly, they serve to model the problem, and secondly, they support the mathematical activity that enables the children to solve the problem. Woleck (2001) highlighted the importance of listening to pupils' explanations of their drawings in order to understand the mathematical activity they are performing. Saundry and Nicol (2006) categorize the different functions of drawings in the problem-solving process as follows: they are used as virtual manipulatives, as a support system, as sophisticated representations, and as images. Starting from the dual (mathematical and textual) nature of arithmetic problem solving, Vicente, Orrantia and Verschaffel (2008) concluded that among children aged 8-10 years, mathematical drawings increased the number of correct answers, whereas situational drawings had no effect on the solving of arithmetic problems involving two operations.

In our opinion, these authors claim that drawing is a counting tool, while leaving implicit the issue as to whether this depends on the way children have been taught. Consequently, the design of our experimental situation involved two groups with a different didactic contract. The goal was to determine the effect of this contract on the types of representations, procedures, arguments and concepts which children use when they individually solve distribution problems.

3. THEORETICAL AND METHODOLOGICAL FOUNDATIONS

In addressing the first research question we did not start from any pre-established theoretical framework, but rather we focused on the systematic collection and analysis of data. Thus, instead of postulating the theoretical categories at the outset, we allowed generalizations to emerge from the data themselves. This approach is consistent with the grounded theory (Glaser and Strauss, 1967).

In order to answer the second research question we made use of some of the theoretical constructs proposed by the onto-semiotic approach (OSA) to cognition and mathematics teaching. In this framework (Font, Godino & Gallardo, 2013), *being* a mathematical object is equivalent to *being involved* somehow in mathematical practices. The expression *mathematical object* is a metaphor that projects some features found in the source domain (physical reality) to the target domain (mathematics). The authors highlight the possibility that physical objects can be separated from other “objects”. Therefore, in principle, all we can "separate" and "individualize" in mathematics is considered an object, for example, a concept, a property, a representation, a procedure, etc. This is an ontological metaphor that allows us to pick up parts of our experience and treat them as discrete entities or substances of a uniform kind. An analysis of mathematical activity reveals a first type of object that is involved in mathematical practices: problems, concepts (understood as implicit or explicit definitions), propositions, procedures and arguments, that is, what OSA refers to as *primary mathematical objects*.

In some studies conducted within the framework of OSA (e.g., Godino, Font, Wilhelmi & Lurduy, 2011; Font, Godino & Gallardo, 2013) with the aim of examining the mathematical output of pupils, the research process begins by analysing mathematical practices and then moves on to consider the primary mathematical objects and processes that are activated within these practices.

In the present study, we have adapted the framework of OSA. Consequently, the practices performed by pupils we are going to analyse are, firstly, their reading of the arithmetic problem that has been set and, secondly, their production of a written answer in response. For reasons of space, and given that we are dealing with a single problem, the paper does not provide an analysis of processes, but rather focuses on the primary mathematical objects that are activated during the practices involved. If one considers the primary mathematical objects that are activated in performing and evaluating the practice that enables a problem to be resolved, what one sees is the use of *representations* (symbolic, etc.). These representations are the ostensive part of a series of *concepts (*understood as definitions*)*, *propositions* and *procedures* that are involved in the development of *arguments* used to decide whether or not the practice carried out is satisfactory. Thus, when a pupil performs and evaluates a mathematical practice, he/she activates a cluster of objects formed by problem situations, representations, concepts, propositions, procedures and arguments, named *cognitive configuration of mathematical primary objects*. Primary objects are related to one another and form configurations, which can be defined as networks of objects that are involved in, and emerge from, systems of practices. These configurations may be epistemic (institutional objects) or cognitive (personal objects).

Our research is an exploratory study that involved the observation of both quantitative variables (how well the problem was solved: correct and incorrect answers) and qualitative variables (type of solution offered by the pupils). For the qualitative analysis of each pupil's answer we made use of the cognitive configuration proposed by OSA. In order to move from the level of individual analysis to a global analysis we made use of three tables for organizing and quantifying qualitative data. The columns in our table are labelled accordingly with three primary mathematical objects and the different kinds of answers provided by pupils are represented in the table' rows.

4. STUDY DESIGN

The study design comprised four stages. The sample selected in the first stage consisted of 21 year-two primary pupils (7-8 years) from a school in Barcelona (Catalonia, Spain), who had never solve partitioning problems before, and whose only experience with problem solving was related to the application of the algorithms for addition and subtraction in problems with an extra-mathematical context (e.g., with statements such as "if John eats 11 almonds for breakfast and 12 more in the afternoon; how many almonds does he eat a day?).

The first stage involved the design of a mathematical task to be solved individually and in written form. The task had the following characteristics: 1) the problem statement was verbal and was not accompanied by any kind of drawing, not even one for aesthetics purposes. The text, though, does not evoke an explicit episodic model, apart from suggesting that pupils are the protagonist of the action expressed in the statement; 2) the problem has an arithmetic distribution in which the whole has to be divided up into discrete sets of various elements, the cardinal number of which may be different; 3) the problem presents an open situation; and 4) children can solve the problem with the existing knowledge they already posses. The problem was stated as follows: "If you have 18 wheels, how many toys with wheels can you have?"

There is a long tradition of this kind of problems in ancient mathematics. The problem is related to the following Diophantine equation: $ax + by + cz + \ldots = k$. In this case, $k = 18$ and the parameters express the number of wheels of a known vehicle (toy). In order to pose a very open problem situation, though, the parameters are not given at the beginning and the choice is left to the child.

As is usual in research on distribution problems of this kind, we chose a number (18) which had several divisors and which was suitable for the children's age; thereby enabling a variety of solutions to the problem. The open problem was selected, among other reasons, because it allowed children to distinguish two separate situations that involved multiplication (namely, the equal partitioning case in which 18 is decomposed multiplicatively (e.g., as 6x3), and an unequal partitioning case which involves repeated groups as part of the total (e.g., as 4x4+2). Moreover, the use of the word "toys" to avoid specifying the particular types of toy is a conscious genuine choice, which allowed us to observe how many children responded by providing the cardinal number corresponding to the set of toys they can make instead of naming the possible set of toys they can make.

This task was presented as follows. First, the problem statement was read out loud and then the children proceeded to solve it, using pencil and paper, during a class that lasted for one hour. When they had finished the task the teacher said to each of them individually: "Explain to me what you've done". The teacher audio recorded each child's response, a process which took around one minute per child (the total time spent was 26 minutes). Their responses were subsequently transcribed by the researcher. It should be noted that the teacher was the class tutor, and the researcher had no relationship with the children. In the second stage we studied the representations produced by the pupils and classified them into two types. Then we drew up a cognitive configuration for each pupil and conducted the global analysis, which was represented in a table.

In the third stage, after analysing the pupils' written answers and producing the configuration of primary objects, we interviewed the children so that they could explain the problem-solving process and the drawings they had used. The aim of this interview was to compare their oral responses with the information obtained from the analysis of written answers. The interview was semi-structured around the following three issues: (1) whether the children thought theirs was the only solution to the problem; (2) whether they thought the problem could be solved in another way; and (3) whether they felt eager to try to solve the problem in as many different ways as possible. These interviews were conducted by the researcher the week after the pupils had given their written answers to the problem. They were audio recorded and transcribed by the researcher. The interviews were conducted over a two-day period and lasted between 5 and 20 minutes, depending on how talkative the children were. As there was one-week delay between doing the tasks and being interviewed, and in order to ensure that children could recall their answers, at the start of each interview the researcher gave children their written answers, along with a pencil and paper in case they needed to develop or improve the answer given initially.

In the fourth stage, the experiment (stages 1-3) was repeated with another class of 22 children from another school in Barcelona. These students had a different didactic contract from the one operating in the first class. The aim here was to show the effect of this contract on the types of representations, procedures, arguments and concepts used by pupils in reaching their solution. In order to determine the influence that class membership might have had on the pupils' drawings, prior to the data collection process in both cases, we analysed the type of tasks children were used to performing, and we also interviewed the teacher of each class so as to gather his/her opinion about why the pupils had used one kind of representation or another. The children in the first group were not used to solving problems in their mathematics class, especially not the type of distribution problem they had been presented with here. The teacher in this class only gave them exercises based on mathematical content that had already been explained to them. The second group of children were quite used to being exposed to the problem-solving methodology in their mathematics class and about six months earlier they had even solved open-ended partitioning problems such as the one presented here. It should be noticed, though, that in that occasions the partitioning problems were resolved in groups of four. Their teacher let us know that she had not organised a group discussion to examine the solution of the problem, instead she had commented it with each group. She had not devoted much time to this task; she simply accepted the possibility that the problem had different solutions if children mentioned it when she encouraged them to look for more than one solution.

Pantziara, Gagatsis & Elia (2009) defined the problem we proposed in both groups as a 'non-routine' problem, especially in the first group. It is those whose solution does not appear immediately and the resolution of which does not involve the application of a procedure previously presented in class.

5. RESPONSE TO THE FIRST RESEARCH QUESTION

In order to answer the first question, it was necessary to conduct a global analysis of the drawings used by the children in the two classes.

5.1 Data analysis

We first classified the representations according to the types of flat projection the children used to draw the three-dimensional objects (toys). These ranged from flat orthogonal projections to flat projections in perspective.

Type of flat projection used:

Children's representation of the flat projection can be grouped into four categories:

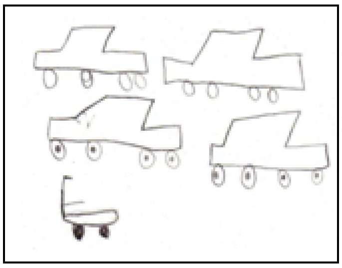
1) Iconic representation in the form of a flat orthogonal projection using a side view. Case 1.1: All the represented objects are drawn (see Figure 1A). Case 1.2: Not all the toys are drawn, but relevant numerical information is included to indicate the number of toys represented.

2) Iconic representation in the form of a flat orthogonal projection, using a side, front or bird's eye view, along with numerical symbols that provide relevant information. Case 2.1: The numerical information indicates the number of wheels on each toy. The pupil has only drawn the wheels that can be seen in

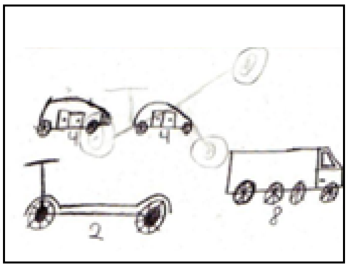
the side view (see Figure 1B). Case 2.2: The numerical information indicates the number of wheels on each toy. The pupil draws a bird's eye view of the toy without drawing any wheels (see Figure 1C).

3) Iconic representation that shows a degree of perspective.

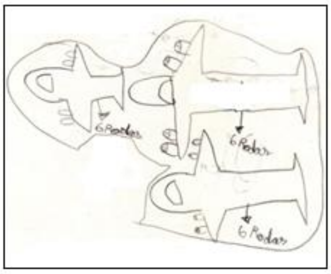
4) Iconic representation in perspective.



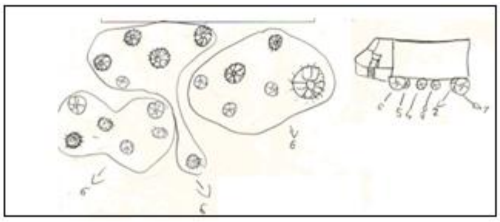
A. Representations used by pupil 6A.



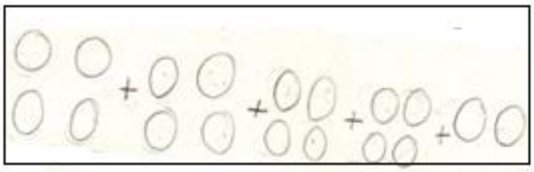
B. Representations used by pupil 20A



C. Representations used by pupil 19B



D. Representations used by pupil 5B.



E. Representations used by pupil 22B.

Figure 1. Some children's drawings

Combination of representations used:

The second classification of the children's drawings looked at how they combined different representations in order to depict the representation by extension (list of members) of the set solution: iconic, verbal, numerical and symbolic representations.

1. All the toys and all the wheels are drawn.

Case 1.1: All the toys are drawn and all their wheels are shown (iconic representation). See Figure 1A.

Case 1.2: All the toys are drawn with all their wheels, along with the name of the object: car, bicycle, car, skates with wheels, etc. (iconic and verbal representation).

Case 1.3: All the toys are drawn with all their wheels, along with the number of wheels (iconic and numerical representation).

Case 1.4: All the toys are drawn with all their wheels, along with the name of the object (car, skate, scooter, etc.) and the number of wheels (iconic, verbal and numerical representation).

2. All the toys but not all the wheels are drawn.

Case 2.1: All the toys are drawn but only some of the wheels. The total number of wheels for each toy is indicated by a number (iconic and numerical representation). See Figure 1B.

Case 2.2: All the toys but none of the wheels are drawn. The total number of wheels for each toy (6 wheels) is indicated by a number (iconic and numerical representation). See Figure 1C.

3. All the wheels but not all the toys are drawn, the extreme case being that no toy is drawn.

Case 3.1: The child draws a set formed by subsets. Each subset contains wheels and the corresponding cardinal number is shown. The picture also includes a toy to illustrate the process of abstraction used in drawing the subsets (iconic and numerical (cardinal) representation, along with a Venn diagram). See Figure 1D.

Case 3.2: The child draws a set formed by subsets. Each subset contains wheels and includes the name of the toy that has given rise to the subset of wheels (motorbike, bicycle, airplane, helicopter, tractor, car, etc.). The drawing also includes the ordinal number used by the pupil to count the number of subsets. Note that the picture does not include a cardinal number indicating the number of wheels in each subset (iconic, verbal and numerical (ordinal) representation, along with a Venn diagram).

Case 3.3: The child draws a set formed by subsets. Each subset contains wheels and the cardinal number of wheels in each subset is also shown (iconic and numerical representation, along with a Venn diagram).

Case 3.4: The child draws a set of wheels formed by subsets that are separated by the plus sign to indicate their union. Note that no Venn diagram or numbers are used (iconic and symbolic representation), see Figure 1E.

5.2 Discussion

Several studies (e.g., Uesaka, Manalo & Ichikawa, 2007) show quite clearly that the construction of an appropriate representation (mentally or externally on paper) is a factor related to performance in problem solving. If we look at the drawings produced by the pupils in both classes, we see that all children four of them needed to make an iconic representation of the set of toys which constituted their answer to the problem set. Hegarty and Kozhevnikov (1999) found that it was possible to differentiate reliably between schematic and pictorial representations made by young children. According to this classification, the drawings made by pupils in Group A tend to be more pictorial and those made by pupils in Group B are more schematic, and occasionally they can be regarded to as diagrams. For example, two pupils in group B (students (B and 4B) provide a tree-diagram representation and 6 pupils produce a Venn diagram. These two representations are categorised as hierarchy type diagrams and part-whole type diagrams respectively (Novick, 2006; Fagnant and Vlassis, 2013).

Woleck (2001) states that representations serve as a cognitive bridge between the concrete and the abstract. The present study corroborates this idea, but also provides greater detail about the different levels of abstraction that can be inferred from the drawings. At one extreme there are children who draw all the toys with all their wheels shown (e.g., Figure 1A), while at the other extreme some children draw various subsets of wheels which are assumed to correspond to a toy that is not explicitly shown at any point (Figure 1E).

The type of flat projection drawings made by the pupils ranged from flat orthogonal projections (side, front or bird's eye view) to flat projections in perspective. This illustrates that the shift from an iconic representation without perspective to one that includes perspective is a complex process at these ages, with numerical symbols playing an important role in completing the information provided when no perspective is represented (e.g., Figure 1B). These types of representation, which constitute a bridge between the iconic flat representations with and without perspective, are quite commonly used by young children when they solve arithmetic problems (e.g. Rubin, Storeygard & Koile, 2013).

Woleck (2001) notes that if students need to draw multiple copies of the same object, they often begin to simplify the drawing, leaving out unnecessary features that might be important in an artistic context, but become cumbersome in a mathematical context. Drawings made by the children in our study corroborate this finding, as the subsequent reproductions of the same object contain fewer artistic details and decrease in size.

The drawings we categorised as belonging to case 1.2., that is, those in which students produce a picture in which not all the toys are drawn, but relevant numerical information is included to indicate the number of toys represented (see figure 2) is, to some extent, related to the so-called *unitizing* process. This means that the representation evokes a group but also that the group is formed by 3 objects. Fosnot and Dolk (2001) examine the role drawings can play in students' learning of multiplication and division. In multiplication, students need to be able to count groups, each of which contains multiple objects. This need requires students to make a major shift in perspective from looking at individual objects to focusing on the group of objects as an individual entity (unitizing).

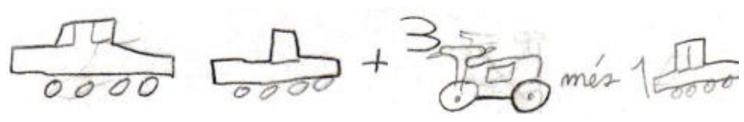
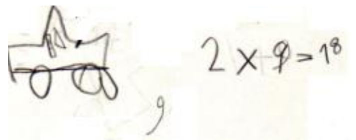
	
A. Representations used by pupil 11A	B. Representations used by pupil 11B

Figure 2. Representations illustrating the unitizing process

The variety of drawings produced by the children in our study also supports the findings of Smith (2003), who studied the role of pupils' idiosyncrasies in relation to the context of the problem. Here we observed that the type of drawing was determined by both personal and class-related aspects (didactic contract). The latter can be inferred from comparing the drawings of the two classes. For example, the use of Venn diagrams to represent sets of wheels only appears in the second class. After analysing the tasks habitually performed in each class and interviewing the respective teachers we concluded that the abovementioned aspects are related to the fact that the pupils: (1) are used to solving problems in class that are more than just applied exercises, as well as to the style of teaching used in this resolution; and (2) have previously solved problems in which the context requires the distribution of the whole into discrete sets of various elements whose cardinal number may also be different.

Our data revealed that solving the problem was a harder tasks for the students in group A (33,4%) compared to the students in group B (72,8%). Moreover, four of the students who did not use a drawing to resolve the problem belong to group B (2B, 7B, 16B and 20B). These results coincide with the studies that state that using drawings as a strategy to model a problem is important if the resolution is difficult for the students but when it is not, the students do not find it necessary to produce a drawing (Smith, 2003; Vicente, Orrantia & Verschaffel, 2008; Woleck, 2001). We would conclude that the difference in the nature of the didactic contract in the two groups explains why the students in group B have fewer difficulties to resolve the set problem.

Our findings also confirm that pupils use drawings as a counting tool, as highlighted by previous studies (Saundry and Nicol, 2006). This usage is implicit in all the representations, and is more explicit in some of them (e.g., Figures 1B, 1C and 1D). To conclude, we should mention that the analysis of each pupil's drawings reveals that those drawings are the result of, and simultaneously enable, different mathematical activity. Consequently, we could argue that the representations used by children are the tip of an iceberg that also comprises calculation procedures, arguments, etc. The analysis of this mathematical activity is the aim of the second research question, and is examined in the next section.

6. RESPONSE TO THE SECOND RESEARCH QUESTION

Two types of analysis were performed for each of the two classes studied (group A and group B) in order to address the second research question (i.e. which procedures, arguments and concepts are associated with the representations used by 7-8 year olds when they individually solve distribution problems in an extra-mathematical context?). We first carried out an individual analysis of each case, and then we conducted a global analysis of all the pupils' output.

For the first analysis the written output of each pupil (e.g., that of pupil 15 in group A, as shown in figure 3) was taken as data, while the problem set and the answer provided by each child were considered as the mathematical practice that had been performed. In accordance with the work of Godino, Font, Wilhelmi & Lurduy (2011), we used the tool known as the cognitive configuration of mathematical primary objects (see, Table 1) in order to determine which part of the activity was associated with children's mathematical practice. In other words, we analysed the following primary objects of the configuration: 1) representations; 2) concepts; 3) properties; 4) procedures; and 5) arguments.

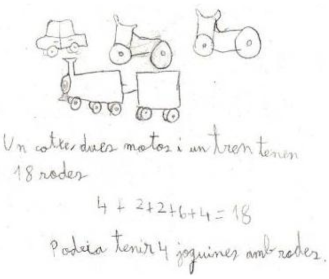
Problem situation	If you have 18 wheels, how many toys with wheels can you have?
Representation	• **Iconic** with perspective • **Symbolic**: - Verbal (one, four, two, ten); - Numbers (4, 2, 6, 18); - Signs (+, =) 
Concepts	• Addition (previous) • Implicit terms of the sum (addends and results) • Number (previous) • Subtraction (implicit) • Set • Elements of a set
Properties	• A number can be broken down into a sum of smaller numbers (applied here to 10 and 18)
Procedures	• Combination of numbers to make 18 • Adding and subtracting (mentally) • Determining a set by extension
Arguments	• **Explicit argument**: I could have four toys with wheels (to make 18) • **Graphical argument**: draws four toys • **Verbal argument**: describes the elements of the set (1 car with four wheels, 1 motorbike with two, another motorbike with two and a train with ten wheels) • **Numerical/written argument**: 4 + 2 + 2 + 6 + 4 = 18

Table 1. Cognitive Configuration of primary objects in the answer given by pupil 15A

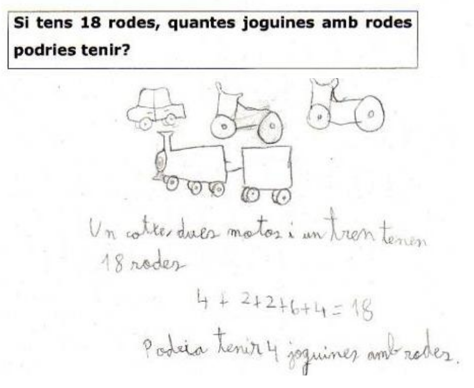
6.1 Analysis of data from group A

We will now discuss two significant cases to provide an example of how we carried out our analysis. The first example is the case of pupil 15A, which will illustrate the whole process of analysis used. For sake of brevity, we will only provide a summary of the second case, that of pupil 20A.

The case of pupil 15A

Pupil 15A solves the problem well and gives the cardinal number corresponding to one of the possible sets, stating: "I could have four toys with wheels". We consider that the possibility of more than one answer can be intuited from this, since the pupil uses the conditional verb form "could". The first thing to note is the richness of this pupil's drawings. She begins with an iconic representation of the toys in perspective (Figure 3) and then translates this into a representation that is both symbolic/numerical (4+2+2+6+4=18) and verbal (a car, two motorbikes and a train have 18 wheels).

With regard to concepts, this pupil is able to break down the set of wheels (18) into parts or subsets (she draws a car with four wheels, two motorbikes with two wheels each, and a train with ten wheels). She is then able to treat each of these subsets as an element (a toy) within a new set (the set of toys). Finally, she makes an implicit distinction between the set and the cardinal number of objects in it, since her answer refers to the cardinal number of toys in each set.

<table>
<tr><th>Written output of pupil 15A</th><th>Verbal output of pupil 15A</th></tr>
<tr><td>

If you have 18 wheels, how many toys with wheels can you have? One car two motorbikes and a train. We have 18 wheels. I could have four toys with wheels.</td><td>I've drawn a car that has four wheels, a motorbike with two, another motorbike that has two, and a train with ten wheels. And here I've explained what I've drawn and how many wheels they have between them. And here I've added up the numbers and put the answer.</td></tr>
</table>

Figure 3. Data from pupil 15A

It should be noted that this pupil applies the mathematical property 'a number can be broken down into the sum of smaller numbers' in order to break down 18 (into three different addends) and then 10 (into two different addends): 4 + 2 + 2 + 6 + 4. We consider that this pupil is aware of how to apply this property, since she writes (6 + 4) and draws a train (6 wheels) pulling a wagon (4 wheels), even though in her verbal response she refers to a train with ten wheels. With respect to procedures, she uses the abovementioned property to break down the number 18. We consider that she starts with a number (she draws a car with four wheels) and then adds another number (she draws a motorbike with two wheels). As the sum is less than 18, she introduces another addend (a motorbike with two wheels), and since the sum is still less than 18 she then introduces a further addend (a train with ten wheels). At the same time she uses an iconic approach to determine the set by extension.

Finally, the explicit reasoning of her demonstrative argument ("I could have four toys with wheels") is justified by the ostensive presentation of the set (iconic representation and verbal description), which she checks numerically in writing (4+2+2+6+4=18). She is aware of this as she says: "…and here I've added up the numbers…"

Given that this pupil's written argument includes the expression "could have" we consider that she knows that the problem could have more than one answer. The interview conducted with her in stage three (Figure 4) confirmed this hypothesis, and means that in the table 2 she must be located in the category 'more than one answer (to the problem) is intuited'.

Researcher. *Can you explain to me what you did first?*

Pupil. *I thought about it and then I did the picture.*

R. *Which picture did you do first?*

P. *I started with the car, then a motorbike, another motorbike, and then the train.*

R. *And how did you come up with this great idea? Why did you draw a car first and then a motorbike...? How did you do it? What were you thinking of every time you drew one of the things?*

P. *Well, first I thought about it, and about some toys that I liked, and then I looked at how many wheels they had and I got to 18. Then I drew the picture.*

R. *And how did you come up with this lovely train?*

P. *Well, I don't know...*

R. *How did you do it? You drew the car, and what did you think?*

P. *I thought, the car has four wheels, and then I drew a motorbike and added two, to make six. And then another motorbike, that's two more wheels, which makes eight. And then the train has six wheels here and four here, which makes 18 wheels.*

R. *And how did you come up with this train joined up like that?*

P. *Why…?*

R. *Up to here you had eight wheels, right? So what did you think of next?*

P. *Well, that six plus four makes ten, so I drew the train.*

R. *You knew that you were missing ten wheels here and you decided to draw a train with ten? And after that what did you do?*

P. *Well, a car, two motorbikes and a train have 18 wheels. After that I added up the numbers: 4+2+2+6+4=18*

R. *And then what did you put?*

P. *That I could have four toys with wheels.*

R. *Do you think there could be more answers, or is this the only one?*

P. *More answers.*

R. *Is that why you wrote 'I could have…'?*

P. *Yes.*

R. *So why didn't you give other answers?*

P. *Well, I don't know...because I didn't want to.*

R. *Do you think you could give me another answer? Look, take this piece of paper in case you want to give me another answer.*

(The pupil begins to draw a second answer that is different to her previous one)

Figure 4. Interview with pupil 15A.

The case of pupil 20A

Pupil 20A also solves the problem implicitly by drawing two cars, a lorry and a scooter (Figure 5). He also gives a verbal response, stating the cardinal number corresponding to each of the subsets (2 cars, 1 scooter and 1 lorry). We believe that the type of representation he uses is also significant. Although his drawings are not in perspective he includes numerical symbols to represent the total number of wheels on each toy. The only conversion he makes is from the iconic and numerical representation of the set of toys to a verbal and written description of the cardinal number corresponding to the subsets. The interview carried out with this pupil in stage three confirmed that he was aware of using the numbers to represent the total number of wheels. Therefore, in the table 2, this pupil falls into the category 'iconic/flat and symbolic/numerical representation'.

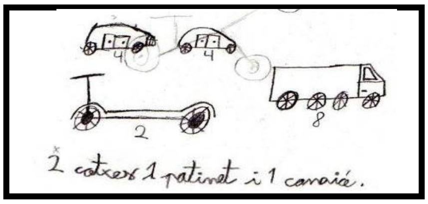


Figure 5. Representations used by pupil 20A.

As regards to concepts, this pupil is able to break down the set of wheels (18) into parts or subsets (he draws two cars with 4 wheels, one scooter with 2 wheels and a lorry with 8 wheels). He is then able to give the cardinal number corresponding to the subsets (2 cars, 1 scooter and 1 lorry). It should also be noted that he implicitly applies the mathematical property 'a number can be broken down into the sum of smaller numbers' in order to break down 18 (into four addends, three of which are different).

With respect to procedures, this pupil implicitly uses the abovementioned property to break down the number 18. After an initial failed attempt with tricycles he starts with a number (he draws two cars with 4 wheels). As the sum is less than 18 he then introduces another addend (a scooter with 2 wheels), and since the sum is still less than 18 he then introduces a further addend (a lorry with 8 wheels). He also uses an iconic approach to determine the set by extension. Finally, the explicit reasoning of his demonstrative argument (2 cars, 1 scooter and 1 lorry) is justified by the ostensive presentation of the set (iconic representation and verbal/written description).

Group A

Table 2 shows the global analysis of the cognitive configurations of all students (groups A and B). The columns are labelled accordingly to the three primary mathematical objects (representation, procedures and properties, and concepts) and the rows provide the different kinds of answers produced by the pupils. The rows are related to the degree of correction of students' answers. Thus, the analysis of pupils' mathematical activity of group A yields two broad categories of answer. The first one refers to those pupils who emphasize the cardinal number of toys in the set. This category can, in turn, be broken down into three sub-categories: 1) those who only give one answer (e.g. pupil 10A: "if I had 18 wheels I would have six toys with wheels"); 2) those whose response suggests more than one answer (e.g. pupil 15A writes "I could have four toys with wheels"); and 3) those who give more than one answer (the only case is that of pupil 18A, who gives four different answers: "… you could have nine 9 motorbikes, or you could have six tricycles…"). The second broad category can also be broken down into three sub-categories: 1) those students who only give one answer, 2) those students whose response suggests more than one answer and 3) those students who give more than one answer. This second category refers to those pupils who (a) emphasize the set of toys; (b) refer to the set only by extension (e.g. pupil 21A: "I've put a car with four wheels, a motorbike with two wheels, a lorry with four wheels and a limousine with eight wheels) or (c) give the cardinal number of toys in each subset (e.g. pupil 6A says "four cars and a bicycle make 18 wheels").

The contents in the columns are related to the mathematical primary objects of the cognitive configuration of each student. The procedures and properties have been grouped into a single row. The arguments provided by students are demonstrative and coincide with their verbal answers. This is why these arguments are shown in the rows (as types of answers).

Kinds of answers / Primary mathematical objects				Representation							Procedures and Properties			Concepts		Total	
				Verbal	Symbolic/ numerical	Iconic			Diagram		Breakdown into product of 2 factors	Breakdown into equal addends	Breakdown into different addends	With multiplication	Without multiplication		
						Perspec.	Flat	Flat-Symbolic	Venn	Tree						A	B
Arguments	Is the cardinal number corresponding to the set of toys	Only one answer		3A, 4A, 10A, 2B, 3B, 4B, 5B, 6B, 7B, 8B, 9B, 10B, 12B, 14B, 15B, 18B 20B	3A, 2B, 3B, 4B 7B, 8B, 11B, 12B, 15B	4A, 10A, 5B, 10B, 14B, 15B	3A, 8B,	11B, 12B	3B, 4B, 5B, 6B, 20B	4B, 8B	2B, 11B	3A, 5B, 7B, 9B, 20B	4A, 10A, 3B, 4B, 6B, 8B, 10B, 12B, 14B, 15B, 18B	3A , 2B, 7B, 11B	4A, 10A , 3B, 4B, 5B, 6B, 8B, 9B, 10B, 12B, 14B, 15B, 18B, 20B	3	15
		More than one answer is intuited		12A, 15A, 17A, 16B	17A , 16B	12A, 15A, 17A						17A, 16B	12A,15A, 16B	17A	12A,15A, 16B	3	1
		More than one answer		18A			18A					18A	18A	18A		1	0
	Is only the set of toys	Only one answer	Gives the set of toys by extension	9A, 16A, 19A, 21A , 21B, 22B	19A	16A , 1B, 17B	2A, 7A, 9A, 19A, 21A, 13B, 21B	8A	22B			7A, 9A, 19A	2A, 8A, 16A, 21A, 1B, 13B, 17B, 21B, 22B	19A	2A, 7A, 8A, 9A, 16A, 21A, 1B, 13B, 17B, 21B, 22B	7	5
			Gives the cardinal number for the subsets of toys	1A, 5A, 6A, 11A, 13A, 14A, 20A		1A, 5A	6A, 13A, 14A	11A, 20A					1A, 5A, 6A, 11A, 13A, 14A, 20A	11A	1A, 5A, 6A, 13A, 14A, 20A	7	0
		More than one answer is intuited	Gives the set of toys by extension	19B	19B	19B						19B	19B		19B	0	1
			Gives the cardinal number for the subsets of toys													0	0
		More than one answer														0	0
Total				36	13	15	13	5	6	2	2	12	32	8	35	21	22

Table 2. The global analysis of the answers given by pupils in groups A and B

6.2 Analysis of group B

We applied the same procedure used with group A. As an example, table 3 shows the individual analysis of the answers given by pupil 2B.

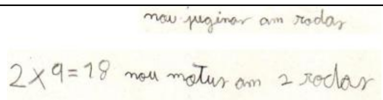
Problem situation	If you have 18 wheels, how many toys with wheels can you have?
Representation	 Nine toys with wheels ; Nine motorbike with 2 wheels • **Symbolic** - Verbal (nine) - Numbers (2, 9, 18) - Signs (x, =)
Concepts	• Multiplication • Explicit terms of the multiplication (factors and product) • Number (previous)
Properties	• 18 is multiple of 2 and 9
Procedures	• Breakdown of a number into two factors • Mental calculation (implicit)
Arguments	• **Explicit argument**: If you have 18 wheels you could have 9 toys with wheels • **Symbolic argument**: 18 can be broken down into the product of two numbers (2 x 9 = 18) and this is contextualized as 9 motorbikes with 2 wheels.

Table 3. Cognitive configuration of primary objects in the answer given by pupil 2B

As regards to concepts, this pupil uses multiplication, and also applies the mathematical property '18 can be broken down into the product of two factors (2 and 9)'. Having broken the number down in this way, the pupil then contextualizes these numbers verbally (9 motorbikes with 2 wheels), and he is one of the only four pupils who did not need to use an iconic representation to resolve the problem. Therefore, this pupil falls into the category 'symbolic/numerical—verbal representation' in the table 2.

It is particularly significant that when he gives his answer he carries out a process of decontextualization, i.e. he says "9 toys with wheels" rather than "9 motorbikes with 2 wheels".

The interview conducted with this pupil in stage three confirmed the hypothesis that he used multiplication directly, without first making use of the property by which the number 18 could be broken down into equal addends. Therefore, this pupil's answer falls into the category 'with the concept of multiplication' in table 2.

As it was expected, the number of students in group B who resolved the problem correctly is higher than those in group A (see table 2). We also need to highlight that, as we said, 4 students in group B (19% of the total) do not produce a drawing, the rest, like all students in group A do need to draw a picture to solve the problem. Important differences between the two groups can be observed with regards to the kind of representation students provide. The category of representations referred to as 'iconic set with or without a Venn diagram' only appears in some of the answers of pupils from group B (see table 2). In general, the representations elaborated by pupils in group B are more abstracts than those created by pupils in group A. This is why the representations that constitute a bridge between flat iconic representations with and without perspective are mostly created by students in group A (see the flat symbolic column in table 2). Contrastively, 50% of the students in group B do not provide any kind of representation or if they do so these are abstract in nature.

6.3 Discussion

The global analysis of the pupils' mathematical productions reveals differences between the two groups in terms of the degree of correctness of their answers and the elements of the configurations of objects that were activated during the problem-solving process. As regards to the former aspect, the correct solution to the problem requires the children to assign the corresponding cardinal number to the set of toys. As it can be seen in Table 3, many more children in group B solved the problem correctly, 72.7% compared to only 33.3% in group A. However, 14.3% of pupils in group A gave more than one correct answer to the problem, illustrating the variety of parameters they assigned, implicitly, to the Diophantine equation ($ax + by + cz + \ldots = 18$) that models the problem. It should also be noted that incorrect answers to the problem were related to the use of a greater number of different types of toys (e.g., 2 cars, 1 scooter and a lorry; as in the answer provided by pupil 20A). This led the children either to give the set by extension (33.3% of pupils in group A and 22.7% in group B) or to give the cardinal number for each subset of toys (33.3% of group A).

Degree of correctness			Frequency Group A	Percentage Group A	Frequency Group B	Percentage Group B
Correct	Just one answer		3	14.3	15	68.2
	More than one answer can be intuited		3	14.3	1	4.5
	More than one answer		1	4.8	0	0
Incorrect	Just one answer	Gives the set of toys by extension	7	33.3	5	22.7
		Gives the cardinal number for the subsets of toys	7	33.3	0	0
	More than one answer can be intuited	Gives the set of toys by extension	0	0	1	4.6
		Gives the cardinal number for the subsets of toys	0	0	0	0
	More than one answer		0	0	0	0
Total			21	100	22	100

Table 3. Frequencies and percentages for the degree of correctness of the problem solutions

The analysis of the different configurations of primary mathematical objects activated during the problem-solving process also reveals differences between the two groups (see Table 2). Firstly, all the pupils in group A produced an iconic representation of their answer (which was not correct in all cases), whereas in group B there were four children (2B, 7B, 16B and 18B) who did not need to rely on an iconic representation in order to give the correct answer to the problem. Secondly, table 3 for group B shows greater variety in the representations and combinations of representations used by the pupils; indeed, two types of representation, namely the use of Venn diagrams (5 pupils) and tree diagrams (2 pupils) only appear in group B. Among the representations used exclusively by children in group A it is worth mentioning what we refer to as 'iconic' and 'symbolic' forms (Figures 1B and 1C), that is, a type of iconic representation that can provide relevant numerical information for solving the problem used by those pupils who draw the toys using a flat representation without perspective.

If we focus solely on the representations produced, as was done in the response to the first research question (section 5.1), it appears that the pupils in group B either do not use drawings or produce drawings that are more abstract than those of group A. However, it should be stressed that the absence of a drawing or its degree of abstraction is not the only aspect to consider when it comes to understanding pupils' mathematical activity. Indeed, we believe that any understanding and evaluation of a pupils' mathematical activity should be based on the analysis of the cognitive configuration that is activated in their answer to the problem. For example, pupils 12A and 3B activated similar cognitive configuration but made use of different representations; pupil 12A drew four toys with four wheels and one with two wheels, while pupil 3B drew four sets of four wheels and one of two wheels. If we focus solely on the drawings we might conclude that the mathematical activity was quite different in these two cases, whereas this difference becomes less significant when we look globally at the cognitive configuration. In table 2, we can observe that pupils 12A and 3B are represented in the same column under the labels "properties and procedures" and "concepts". The argument shown in their drawings (as represented in the rows) is also the same in both cases, except for the fact that pupil 12A says "I could have 5 toys" and pupil 3B says "5 toys with wheels" (this leads us to include student 12A in the rows labelled as "more than one answer can be intuited").

If we have a closer look at the column labelled as "concepts", one clear difference between the groups is the reference to or the explicit use of multiplication. If we focus on those pupils who gave a correct answer to the problem we find that in group B, 19 children didn't use the concept of multiplication and 3 did, while in group A, 5 children used multiplication and 16 didn't. Furthermore, the 6 pupils in group B who failed to solve the problem correctly did not use multiplication, while out of the 14 pupils in group A who didn't solve the problem there were two who did use the concept of multiplication. Therefore, the concepts used by the pupils also illustrate why the unit of analysis of the mathematical activity should be the cognitive configuration and not merely one of its elements. If we look at those pupils who used multiplication, we see that they do not necessarily give the correct answer, or at least not the most general one possible (e.g., pupil 19A), whereas such an answer is given by those pupils who used the sum, a concept that can be considered less abstract than multiplication (e.g., pupil 15A -Figure 3 and Table1- and pupil 3B).

In order to consider that one pupil shows a higher level of abstraction than another, one also needs to take the cognitive configuration into account globally. For example, one could say that the answer given by pupil 2B is more abstract than other answers because: (1) he does not need to use iconic representations; (2) he uses multiplication directly; (3) the fact that he has used multiplication does not lead him to give an answer that involves specifying the type of toy (motorbikes); and (4) his argument does not consist in showing the collection of toys by means of a drawing, but rather he begins with a process of verbal contextualization (9 motorbikes with 2 wheels), followed by a process of decontextualization (9 toys with wheels).

There are four pupils (3A, 17A, 19A and 7B) who separate the set of 18 wheels into discrete sets of equal cardinal numbers and who then begin to solve the problem by using symbolic/numerical written representations (they break down the number 18 into the sum of equal addends). In three of these cases this representation is translated into another symbolic written expression in which they use the concept of multiplication (Figure 6).

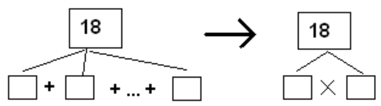


Figure 6. From the breakdown into equal addends to multiplication.

Multiplication was also used by pupil 18A in all four of the different answers he gave (in verbal and iconic form). However, as he does not use written symbolic/numerical representations we assume that he arrived at his answers by using mental arithmetic. From his verbal answers we infer that he makes implicit use of the commutative property (“you can have 6 tricycles with 3 wheels or 3 limousines with 6 wheels and that also makes 18”) and also, unlike the previous three pupils, he does not need to explicitly break down the number 18 into equal addends in order to break it down into the product of two factors. Thus, he does not need to use the sum first in order to arrive at multiplication.

Except for pupil 2B all the children used, implicitly or explicitly, the property of breaking down the number 18 into addends (here we include the extreme case of pupil 18A, who breaks it down into 4x4+2). When the addends are equal this makes it easier to use the concept of multiplication, and also to give the cardinal number corresponding to the set of toys of a certain type (e.g., the answer given by pupil 17A: six tricycles), which implies that the more abstract term 'toy' is not used. By contrast, those pupils who explicitly used the term 'toy' in their answer (e.g., pupil 16B: "I could have 5 toys with wheels") broke down the number 18 into different addends.

This illustrates the close relationship between properties and concepts. The use of a given mathematical property (a particular way of breaking down the number 18) determines the pupils' use of certain mathematical concepts (e.g., sum or multiplication). Furthermore, the fact that they use multiplication — a concept that, in terms of the curriculum, is considered to be more difficult and abstract than is 'sum' — implies, in this case, less abstraction in the answer to the problem. An example of the relationship between representations and concepts can be found in the answers provided by the pupils who use either Venn or tree diagrams. This type of representation leads children to think of the number 18 as a decomposition of its addends instead of configuring it as a decomposition of factors.

Two basic procedures can be observed in the pupils' answers. One is related to the application of the mathematical property or properties that imply breakdown the number 18 into addends. The pupils apply addition and subtraction, and even multiplication, mentally in order to perform this breakdown. The second procedure consists in determining the set by extension (by means of an iconic representation). This second procedure, used by almost all the pupils in the sample, is what enables them to argue, explicitly or implicitly, the reasons for their answer. These arguments are demonstrative and involve the ostensive presentation of the set (the drawings of the toys).

FINAL CONSIDERATION

Several authors have studied the drawings used by children in the mathematics classroom. Pupils' drawings fulfil two basic functions. Firstly, they serve to model the problem, and secondly, they support the mathematical activity that enables the children to solve that same problem. This paper contributes to the literature on the field by providing a detailed description of the mathematical activity employed by children (in terms of practices and configurations of primary mathematical objects). In this context, we understand drawings as a nod in a network of primary objects. We argue that in order to understand mathematic activity supported by drawings we should not focus on the analysis on drawings as objects, instead it is necessary to have a closer look at the whole network activated for their creation, which implies understanding the whole cognitive configuration that made them possible. On doing so, we obtain a wider and more precise view of such an activity, which, in turn, allows us to relativize some of the statements we can formulate when we are solely looking at the drawings.

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